§3. Symmetries

§3.1 Global Symmetries

Consider any infinitesimal transformation of the fields

$$\phi^\ell(x) \longmapsto \phi^\ell(x) + i\varepsilon F^\ell(x)$$

generalized field, can be scalar φ, fermion ψ^α, photon A_μ

spinor index

const.

depends on fields and their derivatives at x

that leaves the action invariant:

$$0 = \delta S = i\varepsilon \int d^4x \, \frac{\delta S[\Phi]}{\delta \phi^\ell(x)} F^\ell(x)$$

"global symmetries"

If we, on the other hand, consider trfs. with ε position dependent,

$$\phi^\ell(x) \longmapsto \phi^\ell(x) + i\varepsilon(x) F^\ell(x)$$

then

$$\delta S = \int d^4x \, J^\mu(x) \frac{\partial \varepsilon(x)}{\partial x^\mu} \qquad (1)$$

$\longrightarrow$ vanishes when ε is const. !

If we take the fields to statisfy the field equations, i.e. S is stationary, then

$$\delta S = \int d^4x \, J^\mu(x) \frac{\partial \varepsilon(x)}{\partial x^\mu} = 0 \quad \forall \varepsilon(x) \ll 1$$

$$\longrightarrow \quad 0 = \frac{\partial J^\mu(x)}{\partial x^\mu}$$

giving $\quad \frac{dQ}{dt} = 0 \quad , \quad Q := \int d^3x \, J^0 \qquad (2)$

$\longrightarrow$ Symmetries imply conservation laws!
"Noether's theorem"

Many symmetries leave $L := \int d^3x \, \mathcal{L}(\vec{x},t)$ itself invariant: translations, rotations, isospin, etc.

$$\longrightarrow \quad \delta S = i \int dt \int d^3x \left[\frac{\delta L[\Phi(t), \dot{\Phi}(t)]}{\delta \Phi^\ell(\vec{x},t)} \varepsilon(t) F^\ell(\vec{x},t) \right.$$

$$\left. + \frac{\delta L[\Phi(t), \dot{\Phi}(t)]}{\delta \dot{\Phi}^\ell(\vec{x},t)} \frac{d}{dt}\left(\varepsilon(t) F^\ell(\vec{x},t)\right) \right]$$

$$\xrightarrow{\varepsilon \text{ const.}} \quad 0 = \int d^3x \left[\frac{\delta L[\Phi(t), \dot{\Phi}(t)]}{\delta \Phi^\ell(\vec{x},t)} F^\ell(\vec{x},t) + \frac{\delta L[\Phi(t), \dot{\Phi}(t)]}{\delta \dot{\Phi}^\ell(\vec{x},t)} \frac{d}{dt} F^\ell(\vec{x},t) \right] \qquad (3)$$

and thus

$$\delta S = i \int dt \int d^3x \; \frac{\delta L[\Phi(t), \dot{\Phi}(t)]}{\delta \dot{\phi}^\ell(\vec{x},t)} \; \dot{\varepsilon}(t) F^\ell(\vec{x},t)$$

from which we can read off (see eq. (1)):

$$\mathcal{J}^0 = i \frac{\delta L}{\delta \dot{\phi}^\ell} F^\ell$$

$$\xrightarrow{(2)} \quad Q = i \int d^3x \; \frac{\delta L}{\delta \dot{\phi}^\ell} F^\ell$$

Using (3), and field eqs.

$$\frac{d}{dt} \frac{\delta L}{\delta \dot{\phi}^\ell} = \frac{\delta L}{\delta \phi^\ell}$$

one can show that indeed $\frac{dQ}{dt} = 0$.

Other symmetries leave even the Lagrangian $\mathcal{L}$ invariant

$$\longrightarrow \delta S = i \int d^4x \left[\frac{\delta \mathcal{L}(\Phi(x), \partial_\mu \Phi(x))}{\delta \phi^\ell(x)} F^\ell(x) \varepsilon(x) \right.$$

$$\left. + \frac{\delta \mathcal{L}(\Phi(x), \partial_\mu \Phi(x))}{\delta(\partial_\mu \phi^\ell(x))} \partial_\mu \left(F^\ell(x) \varepsilon(x) \right) \right]$$

$$\xrightarrow{\varepsilon = \text{const.}} \quad \frac{\delta \mathcal{L}}{\delta \phi^\ell} F^\ell + \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi^\ell)} \partial_\mu F^\ell = 0$$

so for arbitrary ε:

$$\delta S[\Phi] = i \int d^4x \, \frac{\delta \mathcal{L}}{\delta(\partial_\mu \Phi^\ell)} F^\ell(x) \partial_\mu \varepsilon(x)$$

$$\longrightarrow \quad J^\mu = i \frac{\delta \mathcal{L}}{\delta(\partial_\mu \Phi^\ell)} F^\ell$$

$\partial_\mu J^\mu = 0$ when Φ^ℓ satisfy Euler-Lagrange eqs.

<u>Quantization:</u>

Suppose Φ^n is canonical coordinate

$\longrightarrow$ canonical momentum

$$\Pi_n = \frac{\delta \mathcal{L}}{\delta \dot{\Phi}^n} \quad \text{non-vanishing}$$

(if $\frac{\delta \mathcal{L}}{\delta \dot{\Phi}^r} = 0$ for some r then Φ^r is an auxiliary field C^r)

$\rightarrow$ conserved charge (2) can be expressed as

$$Q = i \int d^3x \, \Pi_n(\vec{x},t) F^n(\vec{x},t)$$

$$= i \int d^3x \, \Pi_n(\vec{x},t) F^n[\Phi^\ell(t), \vec{x}]$$

Using canonical commutation relations at equal time

$$[\phi^n(\vec{x},t), \Pi_m(\vec{y},t)]_\mp = i\,\delta^{(3)}(\vec{x}-\vec{y})\,\delta^n{}_m$$

$$[\phi^n(\vec{x},t), \phi^m(\vec{y},t)]_\mp = 0$$

$$[\Pi_n(\vec{x},t), \Pi_m(\vec{y},t)]_\mp = 0$$

one finds (Q is bosonic operator)

$$[Q, \phi^n(\vec{x},t)]_- = F^n(\vec{x},t)$$

$\longrightarrow$ Q is generator of trfs.

$$\phi^\ell(x) \longmapsto \phi^\ell(x) + i\varepsilon F^\ell(x)$$

also
$$[Q, \Pi_n(\vec{x},t)]_- = \int d^3y\, \Pi_m(\vec{y},t)\, \frac{\delta F^m}{\delta \phi^n(\vec{x},t)}$$

Examples:

i) spacetime translations

$$\phi^\ell(x) \longmapsto \phi^\ell(x+\varepsilon) = \phi^\ell(x) + \varepsilon^\mu \partial_\mu \phi^\ell(x) \qquad (*)$$

$$\longrightarrow \quad F^\ell_\mu = -i\partial_\mu \phi^\ell \quad , \mu = 0,1,2,3$$

(4 independent F^ℓ's)

$\rightarrow$ 4 independent conserved currents

$$\partial_\mu T^\mu{}_\nu = 0 \quad , \quad \nu = 0,1,2,3$$

"energy-momentum tensor"

$\rightarrow$ 4 time-independent conserved charges:

$$P_\nu = \int d^3x \, T^0{}_\nu \, , \quad \frac{d}{dt} P_\nu = 0$$

spatial translations leave $L(t)$ invariant

$$\rightarrow \vec{P} = \int d^3x \, \pi_n(\vec{x},t) \vec{\nabla} \phi^n(\vec{x},t) \qquad (4)$$

commutators:

$$[\vec{P}, \phi^n(\vec{x},t)]_- = -i \vec{\nabla} \phi^n(\vec{x},t)$$

$$[\vec{P}, \pi_n(\vec{x},t)]_- = -i \vec{\nabla} \pi_n(\vec{x},t)$$

$\rightarrow$ for any function $G(\pi, \phi)$, we get

$$[\vec{P}, G(x)] = -i \vec{\nabla} G(x)$$

thus we conclude that $\vec{P}$ is generator of space translations

time translations: $P^0 := H$

$$\rightarrow [H, G(\vec{x},t)] = i \dot{G}(\vec{x},t)$$

We can also derive the form of $T^{\mu}{}_{\nu}$

Note that under (*) :

$$\delta S[\phi] = \int d^4x \left(\frac{\delta \mathcal{L}}{\delta \phi^{\ell}} \varepsilon^{\mu} \partial_{\mu} \phi^{\ell} + \frac{\delta \mathcal{L}}{\delta(\partial_{\nu}\phi^{\ell})} \partial_{\nu}[\varepsilon^{\mu}\partial_{\mu}\phi^{\ell}] \right)$$

$$= \int d^4x \left(\frac{\partial \mathcal{L}}{\partial x^{\mu}} \varepsilon^{\mu} + \frac{\delta \mathcal{L}}{\delta(\partial_{\nu}\phi^{\ell})} \partial_{\mu}\phi^{\ell} \partial_{\nu} \varepsilon^{\mu} \right)$$

Integrating by parts,

$$\delta S = \int d^4x \, T^{\nu}{}_{\mu} \partial_{\nu} \varepsilon^{\mu}$$

where $$T^{\nu}{}_{\mu} = \frac{\delta \mathcal{L}}{\delta(\partial_{\nu}\phi^{\ell})} \partial_{\mu}\phi^{\ell} - \delta^{\nu}{}_{\mu} \mathcal{L}$$

$\longrightarrow$ for $\nu=0$, $\mu \neq 0$ integral matches with (4), while for $\nu=0$, $\mu=0$ we get formula for Hamiltonian:

$$H = P_0 = \int d^3x \left[\sum_n \pi_n \dot{\phi}^n - \mathcal{L} \right]$$

ii) invariance under linear coordinate-independent trfs.

$$\phi^n(x) \longmapsto \phi^n(x) + i\varepsilon^a (t_a)^n{}_m \phi^m(x)$$

(**) Hermitian matrices

t_a furnish representation of Lie algebra of symmetry group

$\longrightarrow$ existence of conserved currents $J^\mu{}_a$:

$$\partial_\mu J^\mu{}_a = 0 \longrightarrow T_a = \int d^3x \, J^0{}_a$$

is conserved

when $L(t)$ is invariant under (**)

$$\longrightarrow T_a = -i \int d^3x \, \Pi_n(\vec{x},t)\,(t_a)^n{}_m\,\Phi^m(\vec{x},t)$$

equal time commutation relations give

$$[T_a, \Phi^n(x)] = -(t_a)^n{}_m\,\Phi^m(x)$$

$$[T_a, P_n(x)] = +(t_a)^m{}_n\,P_m(x)$$

$$\longrightarrow [T_a, T_b]_- = i\int d^3x \,\big[-\Pi_m (t_a)^m{}_n (t_b)^n{}_k \Phi^k + \Pi_n (t_b)^n{}_k (t_a)^k{}_m \Phi^m\big]$$

thus $[t_a, t_b]_- = i f_{ab}{}^c\, t_c$ implies

$$[T_a, T_b]_- = i f_{ab}{}^c\, T_c$$

As an illustration, suppose we have two real scalar fields of equal mass

$$\to \mathcal{L} = -\frac{1}{2}\partial_\mu \varphi_1 \partial^\mu \varphi_1 - \frac{1}{2} m \varphi_1^2$$

$$- \frac{1}{2}\partial_\mu \varphi_2 \partial^\mu \varphi_2 - \frac{1}{2} m \varphi_2^2 - \mu(\varphi_1^2 + \varphi_2^2)$$

this is invariant under $SO(2)$:

$$\delta\varphi_1 = -\varepsilon\varphi_2 \, , \qquad \delta\varphi_2 = +\varphi_1$$

$$\to \quad J^\mu = -\varphi_1 \partial^\mu \varphi_2 + \varphi_2 \partial^\mu \varphi_1$$

$$J^0{}_a = -i\,\Pi_n (t_a)^n{}_m \varphi^m$$

equal time commutators:

$$[J^0{}_a(\vec{x},t), \varphi^n(\vec{y},t)] = -\delta^{(3)}(\vec{x}-\vec{y})(t_a)^n{}_m \varphi^m(\vec{x},t)$$

$$[J^0{}_a(\vec{x},t), \Pi_m(\vec{y},t)] = +\delta^{(3)}(\vec{x}-\vec{y})(t_a)^n{}_m \Pi_n(\vec{x},t)$$